A LOWER BOUND ON THE GROWTH OF MINIMAL GRAPHS

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Dedicated to the memory of Peter Duren with gratitude
for his contributions to classical function theory.

Abstract. We show that for minimal graphs in R^3 having 0 boundary values over simpy connected domains, the maximum over circles of radius r must be at least of the order $r^{1/2}$.

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1. Introduction

Let D be an unbounded domain in $\mathbb{R}^2$. We are interested in solutions to the minimal surface equation for the boundary value problem

$$Lu = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1.1}$$

in D with

$$u > 0 \quad \text{in} \quad D, \qquad u = 0 \quad \text{on} \quad \partial D. \tag{1.2}$$

We shall use complex notation $z = x + iy$ for convenience. With $M(r)$ being the maximum value of $u(z)$ on $D \cap \{|z| = r\}$, we have previously studied upper bounds on the growth rate of $M(r)$ under various conditions [3], [9], [10]. We have also obtained some information on the lower bounds.

In [3] we observed the general result

Theorem A. *Suppose D is a domain with $\partial D \neq \emptyset$, and u as in (1.1) and (1.2). Then $u(z)$ has at least logarithmic growth.*

From [10] we have

Theorem B. *Let u satisfy (1.1) and (1.2) with D simply connected and contained in a half plane. Then,*

$$\liminf_{r\to\infty} \frac{M(r)}{r} > 0.$$

In this note we prove the following

Theorem 1. *Suppose that $u(z)$ satisfies (1.1) and (1.2) with D simply connected. Then*

$$\liminf_{r\to\infty} \frac{\log M(r)}{\log r} \geq 1/2. \tag{1.3}$$

The upper half catenoid [2, p.161] shows the necessity of simple connectivity in Theorem 1.

We note that (1.3) with lim sup in place of lim inf follows from [8, Theorem 1]). Also, (1.3) with $1/2$ replaced by $1/\pi$ can be deduced from the work of Miklyukov [5, p.64].

The example given in [9, p. 1085] shows that (1.3) is sharp. (See also [3, p 3391] for related examples.)

2. PRELIMINARIES

Let u be a solution to (1.1) and (1.2) over a simply connected domain D. We shall make use of the parametrization of a surface given by u in isothermal coordinates using Weierstrass functions $(x(\zeta), y(\zeta), U(\zeta))$ with ζ in the right half plane H. Our notation will then be given by

$$f(\zeta) = x(\zeta) + iy(\zeta) \quad \zeta = \sigma + i\tau \in H. \tag{2.1}$$

Then $f(\zeta)$ is univalent and harmonic, and since D is simply connected it can be written in the form

$$f(\zeta) = h(\zeta) + \overline{g(\zeta)} \tag{2.2}$$

where $h(\zeta)$ and $g(\zeta)$ are analytic in H,

$$|h'(\zeta)| > |g'(\zeta)|. \tag{2.3}$$

We dismiss the trivial case $g' \equiv 0$ and may assume for later convenience that $f(0) = 0$. Regarding the height function, we have (cf. [2, §10.2])

$$U(\zeta) = \pm\, 2\Re e\, i \int \sqrt{h'(\zeta)g'(\zeta)}\, d\zeta. \tag{2.4}$$

Now, $z = f(\zeta),\ u(f(\zeta)) = U(\zeta)$ and $U(\zeta)$ is harmonic and positive in H and vanishes on ∂H. Thus, (cf. [7, p. 151]),

$$U(\zeta) = K\, \Re e\, \zeta, \tag{2.5}$$

where K is a positive constant. This with (2.4) gives

$$g'(\zeta) = -\frac{C}{h'(\zeta)}$$

where C is a positive constant. By reparametrizing we may assume that

$$U(\zeta) = 2\Re e\, \zeta \quad \text{and} \quad g'(\zeta) = -1/h'(\zeta), \tag{2.6}$$

and then the *analytic dilatation* [2, p.6] $a(\zeta)$ satisfies

$$a(\zeta) = -1/h'(\zeta)^2. \tag{2.7}$$

Furthermore, from (2.3) we have, in particular, that

$$|h'(\zeta)| = 1/|g'(\zeta)| > 1. \tag{2.8}$$

The strategy will be to analyze $f(\zeta)$ in sectors

$$S_\epsilon = \{(-\pi + \varepsilon)/2 < \arg\zeta < (\pi - \varepsilon)/2\},$$

where $0 < \varepsilon < \pi/2$. We also define, for fixed $\rho > 1$,

$$S_{\varepsilon,n} = S_\varepsilon \cap \{\rho^n \le |\zeta| \le \rho^{n+1}\}\ n = 0, 1, 2, ... \tag{2.9}$$

3. Quasiconformal mappings

We shall have occasion to view the harmonic mapping described in §2 as a quasiconformal mapping. A one to one sense preserving mapping f in a domain D is quasiconformal, if its *complex dilatation* $\delta(\zeta)$ defined by (cf.([2, p. 5]))

$$\delta(\zeta) = \frac{f_{\overline{\zeta}}(\zeta)}{f_\zeta(\zeta)}, \tag{3.1}$$

satisfies

$$\sup_{\zeta\in D} |\delta(\zeta)| < 1. \tag{3.2}$$

Henceforth, we shall refer to $|\delta(\zeta)|$ simply as the dilatation.

The dilatation is a conformal invariant, and the inverse mapping has the same dilatation at corresponding points [1, p. 9].

We shall need a modification of the Ahlfors distortion theorem which requires slight changes in the standard proof [6, pp. 94-97].

In the classical setting we have a simply connected region G with accessible boundary points $Z_1 = X_1 + iY_1$ and $Z_2 = X_2 + iY_2$ Z_1 and Z_2. We assume that $-\infty \le X_1 = \inf \Re e\, z$ for $z \in G$ and $\infty \ge X_2 = \sup \Re e\, z$ for $z \in G$. We consider $z = x + iy$ in G with cross cuts Θ_x separating Z_1 and Z_2 in G (See ([6, pp. 94-95]) for more details). Let $\Theta(x)$ be the length of Θ_x. Let $w(z) = \mu(z) + i\nu(z)$ be a conformal mapping of G onto the strip $\{|\nu| < a/2\}$ such that Z_1 corresponds to $-\infty$ and Z_2 to $+\infty$.

If $\mu_1(x)$ denotes the smallest value on the cross cut and $\mu_2(x)$ the largest, then the classical distortion theorem is as follows.

Theorem C. *If*

$$\int_{x_1}^{x_2} \frac{dx}{\Theta(x)} > 2,$$

then

$$\mu_1(x_2) - \mu_2(x_1) \geq a \int_{x_1}^{x_2} \frac{dx}{\Theta(x)} - 4a.$$

For our purposes, the strip Σ_ε will be the (principal branch) logarithmic image of S_ε in the $w = \mu + i\nu$ plane and G will be the image of a fixed branch of $\log f(S_\varepsilon)$ in the $z = x + iy$ plane with f as in §2. As previously mentioned, we assume for convenience that $f(0) = 0$ so that in G, $\Re e\, z$ extends from $-\infty$ to $+\infty$.

Let $w(z) = \log(f^{-1}(e^z))$ for the principal branch of log which then has the same dilatation as f at corresponding points.

Lemma 1. *With the above notations, let R be a rectangle in the $\mu + i\nu$ plane*

$$R = \Sigma_\varepsilon \cap \ \{\alpha \leq \mu \leq \beta\} \quad 0 < \alpha < \beta,$$

and suppose that $w(z)$ has dilatation less than δ_0 in $w^{-1}(R)$. Then for $x_1 + iy_1$ and $x_2 + iy_2$ in $w^{-1}(R)$ ($x_1 < x_2$) and $a = \pi - \varepsilon$, if

$$\int_{x_1}^{x_2} \frac{dx}{\Theta(x)} > 2\frac{1+\delta_0}{1-\delta_0}, \tag{3.3}$$

then

$$\mu_1(x_2) - \mu_2(x_1) \geq a\frac{(1-\delta_0)}{(1+\delta_0)} \int_{x_1}^{x_2} \frac{dx}{\Theta(x)} - 4a. \tag{3.4}$$

Proof. Our proof follows [6, pp 95-97].

The length of the arc L_x corresponding to $w(\Theta_x)$ is at least $\sqrt{a^2 + \omega(x)^2}$, where $\omega(x) = \mu_2(x) - \mu_1(x)$. Also,

$$L_x \leq \int_{\Theta_x} (|w_z| + |w_{\overline{z}}|)dy \leq \int_{\Theta_x} (|w_z|(1+\delta_0)dy \leq \sqrt{\int_{\Theta_x} dy \int_{\Theta_x} |w_z|^2(1+\delta_0)^2 dy}.$$

Thus,

$$a^2 + \omega(x)^2 \leq \Theta(x) \int_{\Theta_x} |w_z|^2(1+\delta_0)^2 dy.$$

Then,

$$\begin{aligned}
a^2 \int_{x_1}^{x_2} \frac{dx}{\Theta(x)} + \int_{x_1}^{x_2} \frac{\omega(x)^2 dx}{\Theta(x)} &\leq \int_{x_1}^{x2} \int_{\Theta_x} |w_z|^2(1+\delta_0)^2 dydx \\
&= (1+\delta_0)^2 \int_{x_1}^{x_2} \int_{\Theta_x} (|w_z|^2 - |w_{\overline{z}}|^2 + |w_{\overline{z}}|^2)dydx \\
&\leq (1+\delta_0)^2 \int_{x_1}^{x_2} \int_{\Theta_x} (|w_z|^2 - |w_{\overline{z}}|^2 + \frac{\delta_0^2 |w_z|^2}{(|w_z|^2 - |w_{\overline{z}}|^2)}(|w_z|^2 - |w_{\overline{z}}|^2)dydx \\
&\leq (1+\delta_0)^2 \int_{x_1}^{x_2} \int_{\Theta_x} (|w_z|^2 - |w_{\overline{z}}|^2)(1 + \frac{\delta_0^2}{1-\delta_0^2})dydx
\end{aligned}$$

$$= \frac{1+\delta_0}{1-\delta_0}\int_{x_1}^{x_2}\int_{\Theta_x}(|w_z|^2-|w_{\overline{z}}|^2)dydx$$
$$\leq a\frac{(1+\delta_0)}{(1-\delta_0)}(\mu_2(x_2)-\mu_1(x_1))$$
$$= a\frac{(1+\delta_0)}{((1-\delta_0)}(\mu_1(x_2)-\mu_2(x_1)+\omega(x_2)+\omega(x_1)).$$

Summarizing this we have

$$\mu_1(x_2)-\mu_2(x_1)\geq \frac{1-\delta_0}{1+\delta_0}\left(a\int_{x_1}^{x_2}\frac{dx}{\Theta(x)}+\frac{1}{a}\int_{x_1}^{x_2}\frac{\omega(x)^2}{\Theta(x)}dx\right)-\omega(x_1)-\omega(x_2). \tag{3.5}$$

For $x_0\in(x_1,x_2)$, let

$$\lambda(x)=\frac{1}{a}\frac{(1-\delta_0)}{(1+\delta_0)}\int_{x_1}^{x}\frac{\omega(t)^2}{\Theta(t)}dt.$$

Let $m>0$ be fixed and $\mathcal{E}$ be the set of $x>x_0$ such that $\lambda(x)<\omega(x)-m$. Then,

$$a\frac{1+\delta_0)}{(1-\delta_0)}\Theta((x)\frac{d\lambda}{dx}=\omega(x)^2>(\lambda(x)+m)^2,$$

so that

$$\int_{\mathcal{E}}\frac{dx}{\Theta(x)}\leq a\frac{(1+\delta_0)}{(1-\delta_0)}\int_{\mathcal{E}}\frac{d\lambda}{(\lambda+m)^2}\leq\frac{a}{m}\frac{(1+\delta_0)}{(1-\delta_0)}.$$

Similarly, if $\mathcal{F}$ is the set of $x<x_0$ such that $-\lambda(x)<\omega(x)-m$,

$$\int_{\mathcal{F}}\frac{dx}{\Theta(x)}\leq\frac{a}{m}\frac{(1+\delta_0)}{(1-\delta_0)}.$$

Choose x_0 such that

$$\int_{x_1}^{x_0}\frac{dx}{\Theta(x)}=\int_{x_0}^{x_2}\frac{dx}{\Theta(x)}=\frac{1}{2}\int_{x_1}^{x_2}\frac{dx}{\Theta(x)}.$$

Then, if

$$\int_{x_1}^{x_2}\frac{dx}{\Theta(x)}>2\frac{a}{m}\frac{(1+\delta_0)}{(1-\delta_0)},$$

we may take $x_1',\ x_2'$ $(x_1<x_1'<x_2'<x_2)$ such that

$$\int_{x_1}^{x_1'}\frac{dx}{\Theta(x)}=\int_{x_2'}^{x_2}\frac{dx}{\Theta(x)}=\frac{a}{m}\frac{(1+\delta_0)}{(1-\delta_0)}. \tag{3.6}$$

So, there exist $\xi_1\in(x_1,x_1')$ and $\xi_2\in(x_2',x_2)$ such that

$$-\lambda(\xi_1)\geq\omega(\xi_1)-m\ \text{ and }\ \lambda(\xi_2)\geq\omega(\xi_2)-m.$$

Then

$$\frac{1}{a}\frac{(1-\delta_0)}{(1+\delta_0)}\int_{\xi_1}^{\xi_2}\frac{\omega(x)^2}{\Theta(x)}dx=\lambda(\xi_2)-\lambda(\xi_1)\geq\omega(\xi_1)+\omega(\xi_2)-2m.$$

From (3.5)) we then have

$$\mu_1(\xi_2) - \mu_2(\xi_1) \geq a\frac{(1-\delta_0)}{(1+\delta_0)}\int_{\xi_1}^{\xi_2}\frac{dx}{\Theta(x)} + \omega(\xi_1) + \omega(\xi_2) - 2m - \omega(\xi_1) - \omega(\xi_2).$$

From this and (3.6) we deduce that

$$\mu_1(x_2) - \mu_2(x_1) \geq a\frac{(1-\delta_0)}{(1+\delta_0)}\int_{x_1}^{x_2}\frac{dx}{\Theta(x)} - \frac{2a^2}{m} - 2m.$$

With $m = a$ and then $\displaystyle\int_{x_1}^{x_2}\frac{dx}{\Theta(x)} > 2\frac{1+\delta_0}{1-\delta_0}$ we have

$$\mu_1(x_2) - \mu_2(x_1) \geq a\frac{(1-\delta_0)}{(1+\delta_0)}\int_{x_1}^{x_2}\frac{dx}{\Theta(x)} - 4a.$$

□

4. The Parameters

We now select parameters in order to utilize Lemma 1.

To begin with we fix $\varepsilon_1 > 0$ and take $a = \pi - \varepsilon_1$. Next, we fix $0 < \delta_0 < 1/2$ so that $\dfrac{1-\delta_0}{1+\delta_0} > 1 - \varepsilon_1$ and for

$$C_1 = \frac{2\pi}{a(1-\varepsilon_1)} \tag{4.1}$$

define ε_2 by

$$\varepsilon_2 = C_1 - 2. \tag{4.2}$$

We then define

$$C_2 = \exp(\frac{8\pi}{(1-\varepsilon_1)} \tag{4.3}$$

and fix a value ρ in (2.9) large enough so that

$$\rho > e^{5\pi} \quad \text{and} \quad C_2/\rho^{\varepsilon_2} < 1/2. \tag{4.4}$$

The R in Lemma 1 now corresponds to a sequence

$$R_{\varepsilon_1 n} = \Sigma_{\varepsilon_1} \cap \{(n-1)\log\rho \leq \mu \leq n\log\rho\}.$$

Finally, we note that since $\log|h'|$ is a positive harmonic function in H, if M is the maximum of $\log|h'|$ in $S_{\varepsilon_1,n}$ and m the minimum of $\log|h'|$ in $S_{\varepsilon_1,n}$, there exists a constant $K = K(\varepsilon_1, \rho)$ (independent of n) such that $m/M \geq K$ Thus, we may fix a value M_0 for M such that if $\log|h'|$ has maximum greater or equal to M_0, then the minimum of $\log|h'|$ will be greater than $\log(1/\sqrt{\delta_0})$ and hence by (2.7)

$$|\delta(\zeta)| = 1/|h'(\zeta)|^2 < \delta_0 \quad \zeta \in S_{\varepsilon_1,n}.$$

The objective will be to bound $\log|f(\sigma)|$ for large σ, with N chosen so that

$$\rho^{N-1} < \sigma < \rho^{N}.$$

Note that $|f(\sigma)|$ is unbounded. In fact since $U(\sigma) = 2\sigma$ (recall (2.6)), if $|f(\sigma)|$ were bounded, then (1.3) would hold trivially.

5. Proof of Theorem 1

With the conventions in §4, we distinguish two cases.

Case 1. Here we consider the case in which the maximum of $\log|h'|$ is at least M_0 in a given $S_{\varepsilon_1 n}$. For the corresponding $R_{\varepsilon_1 n}$, then $w(z) = \log(f^{-1}(e^z))$ has dilatation less than δ_0 in $w^{-1}(R_{\varepsilon_1 n})$, and in Lemma 1 we take $x_1 + iy_1 = w^{-1}((n-1)\log\rho)$ and $x_2 + iy_2 = w^{-1}(n\log\rho)$. We claim that in this case,

$$\begin{aligned} n\log\rho - (n-1)\log\rho &\geq \mu_1(n\log\rho) - \mu_2((n-1)\log\rho) \\ &\geq a(1-\varepsilon_1)(\frac{1}{2\pi})(\log|f(\rho^n) - \log|f(\rho^{n-1})|) - 4a. \end{aligned} \tag{5.1}$$

In fact, since $\rho > e^{5\pi}$, if (3.3) does not hold, then (5.1) holds automatically. Otherwise (5.1) follows from (3.4). We rewrite (5.1)

$$\log|f(\rho^n) - \log|f(\rho^{n-1})| < \frac{2\pi}{a(1-\varepsilon_1)}\log\rho + \frac{8\pi}{(1-\varepsilon_1)}.$$

so that

$$|f(\rho^n)| < C_2\rho^{C_1}|f(\rho^{n-1})|, \tag{5.2}$$

where C_1 is as in (4.1) and C_2 in (4.3). From (5.2) it follows that

$$\begin{aligned} \frac{|f(\rho^n)|}{\rho^{n(C_1+\varepsilon_2)}} - \frac{|f(\rho^{n-1})|}{\rho^{(n-1)(C_1+\varepsilon_2)}} &< \frac{|f(\rho^n)|}{\rho^{n(C_1+\varepsilon_2)}} - \frac{|f(\rho^{n-1})|}{\rho^{n(C_1+\varepsilon_2)}} \\ (C_2\rho^{C_1} - 1)\frac{|f(\rho^{n-1})|}{\rho^{n(C_1+\varepsilon_2)}} &< \frac{C_2\rho^{C_1}}{\rho^{C_1+\varepsilon_2}}\frac{|f(\rho^{n-1})|}{\rho^{(n-1)(C_1+\varepsilon_2)}} \end{aligned} \tag{5.3}$$

Case 2. If, on the other hand, the maximum of $\log|h'|$ in S_n is less than M_0,

$$|f(\rho^n)| - |f(\rho^{n-1})| \leq \int_{\rho^{n-1}}^{\rho^n}(|h'(t)| + 1/|h'(t)|)dt \leq 2e^{M_0}(\rho^n - \rho^{n-1}). \tag{5.4}$$

From (5.4) we may write

$$\frac{|f(\rho^n))|}{\rho^{n(C_1+\varepsilon_2)}} - \frac{|f(\rho^{n-1}))|}{\rho^{(n-1)(C_1+\varepsilon_2)}} < \frac{|f(\rho^n)|}{\rho^{n(C_1+\varepsilon_2)}} - \frac{|f(\rho^{n-1}))|}{\rho^{n(C_1+\varepsilon_2)}} \leq 2e^M\frac{(\rho^n - \rho^{n-1})}{\rho^{n(C_1+\varepsilon_2)}} \tag{5.5}$$

Combining (5.3) and (5.5) we may write

$$\frac{|f(\rho^n)|}{\rho^{n(C_1+\varepsilon_2)}} - \frac{|f(\rho^{n-1})|}{\rho^{(n-1)(C_1+\varepsilon_2)}}\left(\frac{C_2\rho^{C_1}}{\rho^{C_1+\varepsilon_2}} + \frac{1}{\rho^{C_1+\varepsilon_2}}\right) < \frac{2e^M(1-1/\rho)}{\rho^{n(C_1+\varepsilon_2-1)}}. \tag{5.6}$$

Using (4.4) and summing (5.6) up to $n = N-1$ we have

$$\frac{|f(\rho^{N-1}))|}{\rho^{(N-1)(C_1+\varepsilon_2)}} < |f(1)| + K. \tag{5.7}$$

where $K = \sum_{n=1}^{\infty} 2e^M(1-\rho)/\rho^{n(C_1+\varepsilon_2-1)} < \infty$ since $C_1 > 2$.

To estimate $|f(\sigma)|$ for $\rho^{N-1} < \sigma < \rho^N$, we first consider Case 1. If the condition in (3.3) fails, then

$$\log|f(\sigma)| - \log|f(\rho^{N-1})| \leq 4\pi\frac{1+\delta_0}{1-\delta_0}. \tag{5.8}$$

which with (5.7) gives

$$\log|f(\sigma)| < \log(|f(1)| + K) + (N-1)(C_1+\varepsilon_2)\log\rho + \frac{4\pi}{1-\varepsilon_1}. \tag{5.9}$$

Otherwise, as in (5.4)-(5.7) we have

$$\frac{|f(\sigma))|}{\rho^{N(C_1+\varepsilon_2)}} < |f(1)| + K. \tag{5.10}$$

Comparing (5.9) and (5.10) as before, we deduce that (5.10) holds.

Using the fact that $U(\sigma) = 2\sigma$ (recall (2.6)), $\sigma > \rho^{N-1}$, and $C_1 + \varepsilon_2 = 2 + 2\varepsilon_2$ we have

$$\frac{\log|f(\sigma)|}{\log U(\sigma)} < \frac{\log|f(\sigma)|}{\log(2\rho^{N-1})} < \frac{\log\rho^{N(2+2\varepsilon_2)}}{\log(2\rho^{N-1})} + \frac{\log(|f(1)|+K)}{\log(2\rho^{N-1})}.$$

Since ε_2 can be made arbitrarily small, (1.3) follows. $\square$

References

1. L. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand Mathematical Studies, 1966.
2. P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, 2004.
3. E. Lundberg, A. Weitsman, *On the growth of solutions to the minimal surface equation over domains containing a half plane*, Calc Var. Partial Differential Equations **54** (2015) 3385-3395.
4. O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, Springer-Verlag, 1973.
5. V. Mikljukov, *Some singularities in the behavior of solutions of equations of minimal surface type in unbounded domains*, Math. USSR Sbornik **44** (1983) 61-73.
6. R. Nevanlinna, *Analytic functions*, Springer-Verlag, 1970.
7. M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo (1959).
8. A Weitsman, *On the growth of minimal graphs*, Indiana Univ. Math. J. **54** (2005) 617-625.
9. A. Weitsman *Growth of solutions to the minimal surface equation over domains in a half plane*, Communications in Analysis and Geometry, **13** (2005) 1077-1087.
10. A. Weitsman, *A sharp bound for the growth of minimal graphs*, Computational Methods in Function Theory **21** (2021) 905-914.

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